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All Authors	Shwin Seinn
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Abstract

In this paper, the necessary conditions for a bipartite graph to be Hamiltonian are first discussed. Relating these conditions, the movements of a chess piece called Knight, in the game of chess are discussed. And it is shown that finding a Hamiltonian cycle in a given bipartite graph of 64 vertices is equivalent to finding a reentrant Knight's tour on a 8×8 chessboard. Furthermore, the domination concepts of graphs are discussed. Relating these conditions, the movements of a chess piece called Queen are discussed. And it is shown that finding the independence number and independent domination number of a given graph of 64 vertices are equivalent to finding respectively the maximum number of Queens which control each square and the minimum number of attacking Queens and non-attacking Queens on a 8×8 chessboard.

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Introduction

Graphs are discrete structures consisting of vertices and edges that connect these vertices. Sometimes a graph has the property that its vertex set can be divided into two disjoint subsets such that each edge connects a vertex in one of these subsets to a vertex in the other subset. Such a graph will call bipartite and some results concerning this type of graph were presented in [7]. And this research work is the continuation of [7].

Lecturer, Dr, Department of Mathematics, Kalay University

Since most of us play the game of chess and many real-world problems can be modeled as a graph, part of this work will discuss the movement of a Knight in the game of chess as a graph and how it will relate to some results shown in [7]. Also, another chess piece called a Queen and it is an interesting case for how its movements and the number of Queens will relate to the domination concepts of a graph.

The objective of this research paper is to give the relations between some pieces in the game of chess and some properties of graphs. 5-section are organized in this research paper. The first section describes the basic terminology and notations of graph theory. In the second section, the works of [7] are overviewed. The nature and movement of a piece in the game of chess called Knight are presented in the third section The 8×8 chessboard which can be modeled as a bipartite graph is presented in that section. Also it is declared that finding a Hamiltonian cycle in a given bipartite graph of 64 vertices is equivalent to finding a reentrant Knight's tour in the game of chess. Domination concepts of graphs are studied in the fourth section. At the last section, the nature and movements of another chess piece called Queen are presented. Consequently, it is shown that finding the domination numbers of a given graph of 64 vertices are equivalent to finding the minimum number of attacking Queens and non-attacking Queens on a '8×8' chessboard.

Terminology and Notations

Some basic concepts of graph theory can be seen in [1], [2], [3], [4], [5], [6] and [7]. Other special definitions that are only relevant in certain sections will be given where they are needed. To be discussed for the later sections, we briefly give some terminology and notations of graph.

A **walk W** in a graph G is a finite non-null sequence $v_0e_1v_1e_2v_2 \cdots e_kv_k$, whose terms are alternately vertices and edges such that for $1 \le i \le k$, the ends of e_i are v_{i-1} and v_i. We say that W is a walk from v_0 to v_k or a $\mathbf{v_0v_k}$ **– walk**. The vertices v_0 and v_k are called the **origin** and **terminus** of W, respectively and $v_1, v_2, \cdots, v_{k-1}$ are its **internal vertices**. The integer k, the number of edges in W is the **length of W**. A v_0v_k – walk is said to be **closed** if $v_0 = v_k$. A walk is called a **path** if there are no vertex repetitions. A **Hamiltonian path** is a path that contains all vertices of a graph where the end points (vertices) may be distinct.

A **cycle** in a graph G is a path that begins and ends at the same vertex, i.e., a closed path of non-zero length that does not contain a repeated edge. A **Hamiltonian cycle** is a cycle in a graph G that contains each vertex exactly once except for the starting and ending vertex that appears twice. If a graph G contains a Hamiltonian cycle, then it is said to be **Hamiltonian.**

Hamiltonicity of Bipartite Graphs

This section overviews the work of [7].

Definition 1

A graph G = (V, E) is said to be **bipartite** if the vertex set 'V' of G can be partitioned into two subsets V_1 and V_2 such that each edge connects a vertex in V_1 and a vertex inV_2 or no edges of G join vertices in the same subset.

Definition 2

A **complete bipartite graph** is a graph with vertex set partitioned into a subset V_1 of 'm' elements and a subset V_2 of 'n' elements such that two vertices are connected by an edge if and only if one is in V_1 and the other is in V_2. A complete bipartite graph on 'm' and 'n' vertices is denoted by $K_{m,n}$.

The condition for a bipartite graph to be Hamiltonian is depending on the numbers of vertices in its two partite sets. [7] showed these condition as follows.

Theorem 1

If a bipartite graph G with partite sets V_1 and V_2 is such that $|V_1| \neq |V_2|$, then G is non-Hamiltonian.

Corollary 1

A bipartite graph having odd number of vertices is non-Hamiltonian.

Theorem 2

A complete bipartite graph $K_{m,n}$ is Hamiltonian if and only if $m = n$, for all $m, n \geq 2$.

Knight's Tour Problem

In this section, the nature and movements of a chess piece called Knight in the game of chess are described. And, some observations concerning these movements will show in terms of Hamiltonicity of corresponding bipartite graph.

Definition 3

A **Knight** is a chess piece that can move either two spaces horizontally and one space vertically or one space horizontally and two spaces vertically. That is, a Knight on square (x, y) can move to any eight squares $(x \pm 2, y \pm 1)$, $(x \pm 1, y \pm 2)$; if these squares are on the chessboard (which is normally $'8 \times 8'$).

Proposition 1

The graph representing the legal moves of a Knight on an 'm × n' chessboard, whenever 'm' and 'n' are positive integers is a bipartite graph

Proof:

Let each square of the board be a pair of integer (x, y), where $1 \le x \le m$ and $1 \le y \le n.$ Let A be the set of squares for which x + y is odd and B, the set of squares for which x + y is even. This partition makes the vertex set of the graph representing the legal moves of a Knight on the chessboard into two parts. Since every move of the Knight changes x + y by an odd number either 3 or 1, every edge in this graph joins a vertex in A to a vertex in B. Thus, the graph is bipartite.

Definition 4

A **Knight's tour** is a sequence of legal moves by a Knight starting at some square and visiting each square exactly once. A Knight's tour is called a **reentrant** if there is a legal move that takes the Knight from the last square of the tour back to where the tour began.

From Proposition1 and Corollary1, the following result is obtained.

Result 1

There is no reentrant knight's tour on a $'m \times n'$ chessboard when 'm' and 'n' are both odd.

Proof:

Since there are mn squares on the $m \times n$ chessboard, if both 'm' and 'n' are odd, then there are an odd number of squares. We have known from Proposition 1 that the Knight's tour graph is bipartite and also from Corollary 1, there is no reentrant Knight’s tour on a $m \times n$ chessboard with both 'm' and 'n' odd.

Lemma 1

There is a reentrant Knight's tour on a $'8 \times 8'$ chessboard.

Proof:

We can construct a reentrant Knight's tour on a $'8 \times 8'$ chessboard by starting any square and then always move to a square connected to the fewest numbers of unused square. With this command, we show that there is a reentrant Knight's tour on a $'8 \times 8'$ chessboard as shown in Figure 1.

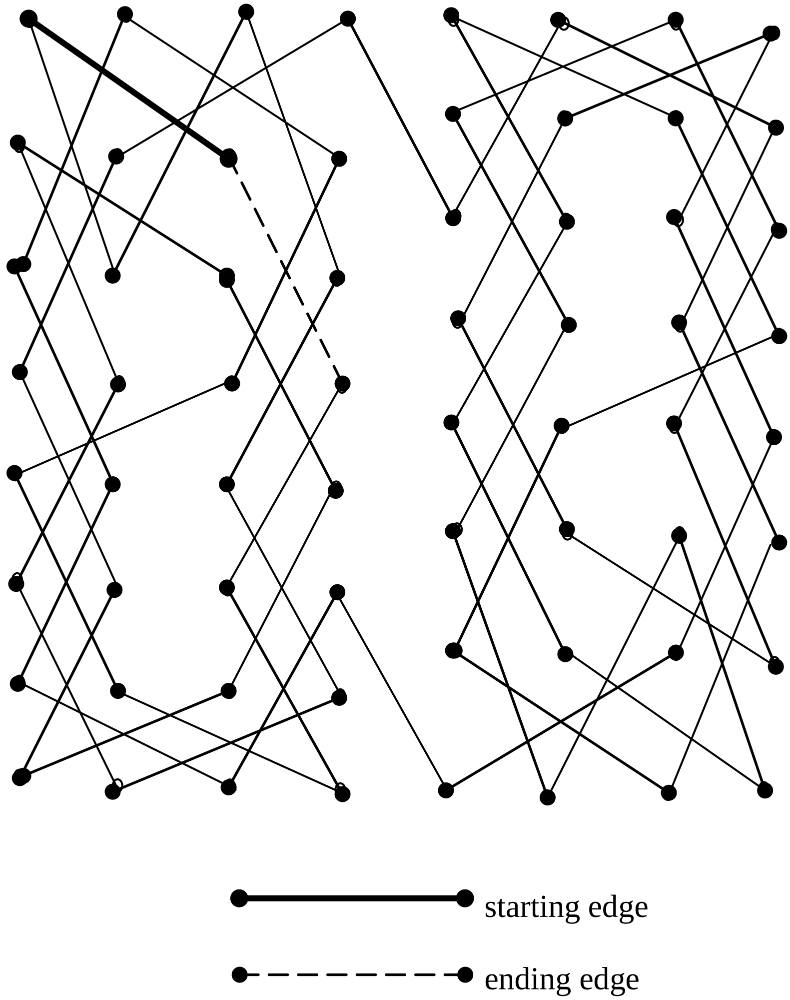

Figure 1

From Proposition 1 and Lemma 1, the following result is obtained.

Result 2

"finding a reentrant Knight's tour in '8×8' chessboard is equivalent to finding a Hamiltonian cycle in a corresponding bipartite graph".

Domination Concepts of Graphs

In this section, the independent set, the dominating set, the independent dominating set, the independence number, the domination number and the independent domination number of a graph are discussed.

Definition 5

An **independent set** of a simple graph G is a set I of vertices of G such that no two vertices in it are adjacent.

Example 1

We will give an independent set of the graph shown in Figure 2.

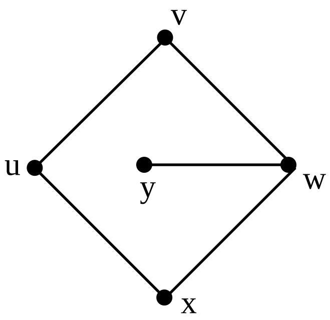

Figure 2

In the graph G of Figure 2, $V = \{u, v, w, x, y\}$. Take $I = \{v, y, x\}$.

It is clear that no two vertices in I are adjacent.

Thus, $I = \{v, y, x\}$ is an independent set of the given graph.

Definition 6

The **independence number** of a simple graph G, denoted by $\alpha(G)$ is the maximum size of an independent set in G.

Example 2

We will give the independence number of the graph shown in Figure 3.

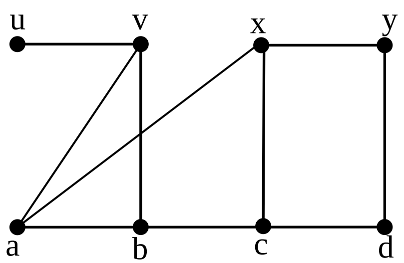

Figure 3

In the graph G of Figure 3,

$V = \{a, b, c, d, u, v, x, y\}$. Take $I = \{u, b, x, d\}$.

It is clear that no two vertices in I are adjacent.

Thus, I is an independent.

Again, take $J = \{v, c, y\}$.

It is clear that no two vertices in J are adjacent.

Thus, J is also an independent.

Again, take $M = \{a, u, y, c\}$.

It is clear that no two vertices in M are adjacent.

Thus, M is also an independent.

We have found that I and M are maximal independent set consisting of four vertices.

Therefore, the independence number of the given graph is 4.

Definition 7

A **dominating set** of a simple graph G is a set S of vertices of G such that every vertex not in S is adjacent to at least one vertex in S. The dominating set with the least number of vertices is called the **minimal dominating set**.

Example 3

We will give two dominating sets and minimal dominating set for the graph shown in Figure 4.

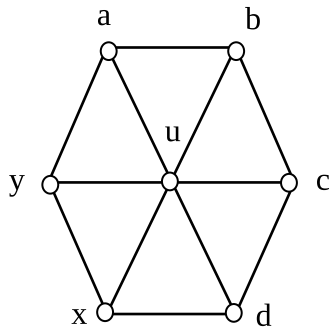

Figure 4

In the graph G of Figure 4,
$V=\{a,b,c,d,x,y,u\}$. Take $S=\{b,y,d\}$.

Then, $V\setminus S=\{a,c,x,u\}$.

Every vertex in V \ S is adjacent to at least one vertex of S.

Thus, $S=\{b,y,d\}$ is a dominating set of the given graph.

Again, if we take $S=\{u\}$, then, $V\setminus S=\{a,b,c,d,x,y\}$.

Every vertex in V \ S is adjacent to the vertex u of S.

Thus, $S=\{u\}$ is a dominating set of the given graph.

Hence, the minimal dominating set of the given graph is $S=\{u\}$.

Definition 8

The **domination number** of a simple graph G, denoted by $\gamma(G)$ is the minimum cardinality among the dominating sets of G.

Example 4

The graph shown in Figure 5 is the Petersen graph. We will find its domination number.

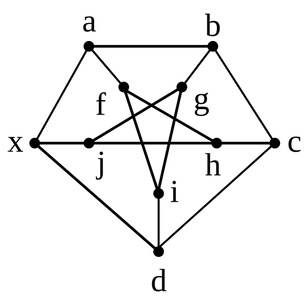

Figure 5

In the Petersen graph G of Figure 5,
$V = \{a, b, c, d, x, f, g, h, i, j\}$. Take $S = \{x, g, h\}$.

Then, $V \setminus S = \{a, b, c, d, f, i, j\}$.

Every vertex in V \ S is adjacent to at least one vertex of S.
Thus, $S = \{x, g, h\}$ is a dominating set of the Petersen graph.

Again, if we take $S = \{f, g, h, i, j\}$, then, $V \setminus S = \{a, b, c, d, x\}$.
Every vertex in V \ S is adjacent to one vertex of S.
Thus, $S = \{f, g, h, i, j\}$ is a dominating set of the Petersen graph.

Thus the minimal dominating set consists of three vertices since each vertex of the Petersen graph dominates three other vertices.

Hence the Peterson graph has domination number 3.

Definition 9

An **independent dominating set** of a simple graph G is a set D of vertices of G such that D is both dominating and independent sets of G.

Example 5

We will give an independent dominating set of the graph shown in Figure 6.

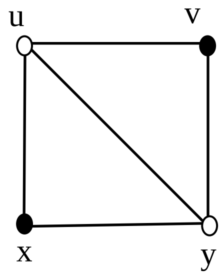

Figure 6

In the graph G of Figure 6, $V = \{u, v, x, y\}$. Let $S = \{x, v\}$.

We find that $S = \{x, v\}$ is both the independent set and the dominating set of the given graph.

Therefore, $S = \{x, v\}$ is the independent dominating set of the given graph.

Definition 10

The **independent domination number** of a simple graph G, denoted by i(G) is the minimum size of an independent dominating set of G.

Example 6

We will find the independent domination number of the graph shown in Figure 7.

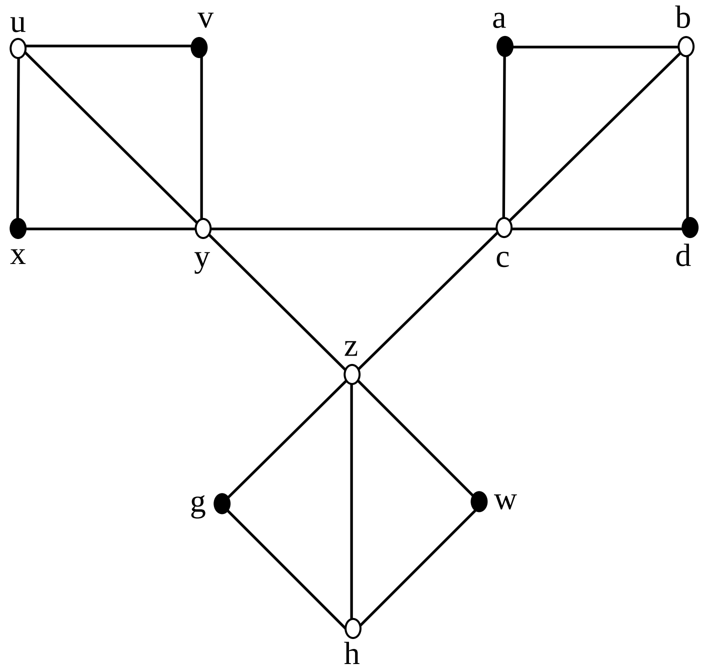

Figure 7

In the graph G of Figure 7, $V = \{a, b, c, d, u, v, x, y, z, g, h\}$. We find that $S = \{x, v, a, d, g, w\}$ is a maximal independent set and a dominating set of the given graph.

We also find that $M = \{u, b, h\}$ is a minimal independent set and a dominating set of the given graph.

Therefore, the independent domination number of the given graph is 3.

Queens Graph

In this section, the applications for domination concepts of graphs will be discussed. Such applications concern with the number of (attacking or non-attacking) queens on a chessboard.

Definition 11

A **Queen** is a chess piece that can move either horizontally, vertically, or diagonally over any number of unoccupied squares.

Definition 12

The 64 squares of a chessboard are the vertices of a graph G and two vertices (squares) are adjacent in G if each square can be reached by a Queen on the other square by a single move. The graph G is referred to as the **Queen graph.**

Example 7

We will find the minimum number of Queens so that each square is controlled by at least one Queen and give the position of such Queens on a '8×8' chessboard.

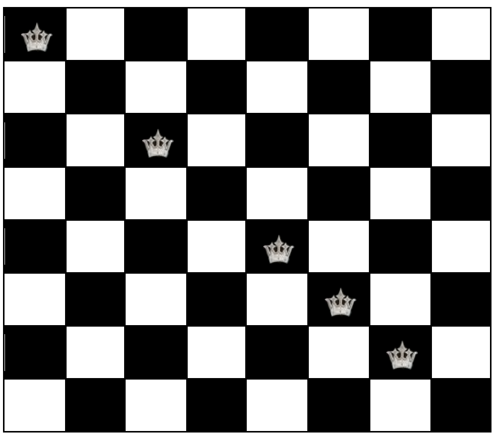

Figure 8

According to Figure 8, we find that the minimum number of Queens so that each square is controlled by at least one Queen is 5.

Example 8

We will find the maximum number of Queens so that each square is controlled by at least one Queen and give the position of such Queens on a $'8\times 8'$ chessboard.

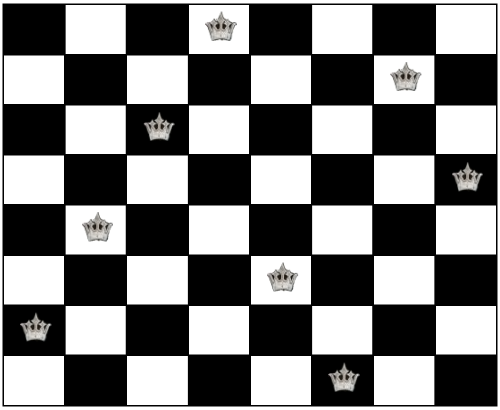

Figure 9

According to Figure 9, each square occupied by a Queen is not adjacent to each other. Thus, the squares as the vertices form a maximal independent set of the corresponding graph. Thus, we find that the maximum number of Queens so that each square is controlled by at least one Queen is 8.

Observation 1

"Finding the maximum number of Queens so that each square is controlled by at least one Queen in a chessboard is equivalent to finding the independence number of the corresponding graph".

Definition 13

Two Queens on a chessboard are **attacking Queens** if the square occupied by one of the Queens can be reached by the other Queen in a single move.

Remarks

On a $'2\times 2'$ chessboard, there is no attacking Queens.
Also, for a $'3\times 3'$ chessboard, there is no attacking Queens.

Example 9

We will find the minimum number of attacking Queens and give the position of such Queens on a $'8\times 8'$ chessboard.

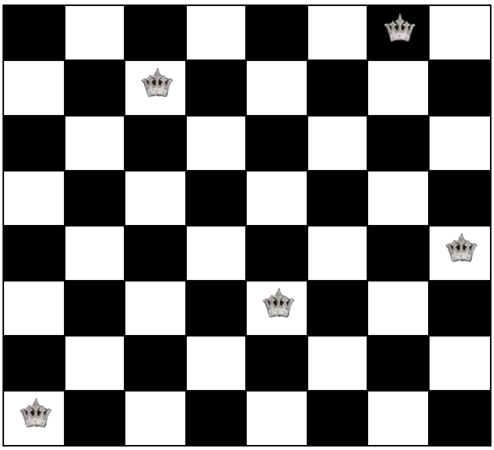

Figure 10

According to Figure 10, each square occupied by a Queen is adjacent to the other square. Thus, the squares as the vertices form a minimal dominating set of the corresponding graph. Thus, we find that the minimum number of attacking Queens on a $'8\times 8'$ chessboard is 5.

Observation 2

"Finding the minimum number of attacking Queens that attack all the squares of a chessboard is equivalent to finding the domination number of the corresponding graph".

Definition 14

Two Queens on a chessboard are **non-attacking Queens** if the square occupied by one of the Queens cannot be reached by the other Queen in a single move.

Example 10

We will find the minimum number of non-attacking Queens and give the position of such Queens on a '8×8' chessboard.

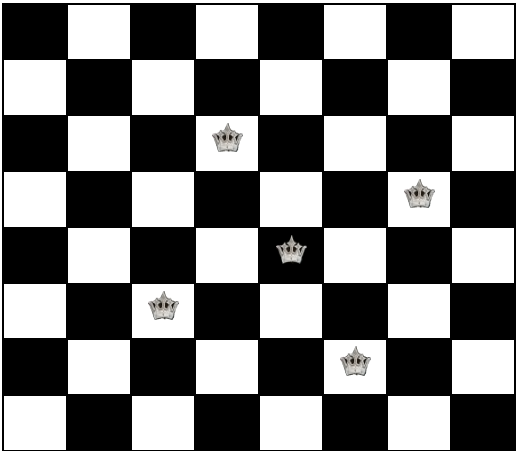

Figure 11

According to Figure 11, each square occupied by a Queen is not adjacent to each other. Thus, the squares as the vertices form an independent set of the corresponding graph.

They also form a minimal dominating set since each square except occupied by a Queen is adjacent to at least one square occupied by a Queen. Thus, we find that the minimum number of non-attacking Queens on a $'8\times8'$ chessboard is 5.

Observation 3

"Finding the minimum number of non-attacking Queens that attack all the squares of a chessboard is equivalent to finding the independent domination number of the corresponding graph".

Conclusion

In this work, we investigated on the relations between some pieces in the game of chess and some properties of graph. Moreover, we explored the Knight's tour problem and Queens graph. It is found that finding a reentrant Knight's tour in the chessboard is equivalent to finding a Hamiltonian cycle in a corresponding bipartite graph. It is also exhibited that finding the minimum number of attacking Queens in the chessboard is equivalent to finding the domination number of a corresponding graph. Moreover, it is found that finding the minimum number of non-attacking Queens on a chessboard is also finding the independent domination number of a corresponding graph.

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